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Applications of Invariant Manifold

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Abstract: **In this paper we discuss invariant manifolds and stability analysis (stable and unstable manifold) associated to the fixed point of a dynamical system. We explain the significance of eigenvalues of Jacobian matrix and we consider some examples. We conclude with applications of these concepts.**

Keywords: **invariant manifolds, dynamical system, Jacobian matrix.**

1. INTRODUCTION

The invariant manifolds shared and played important role in the theory of dynamical systems. They can be used to ease the analysis of complex system. This can contain the transformation of finite dimensional problems into finite dimensional ones. The stable and unstable points in a dynamical system are loosely defined as points which converge to points in forward time, so we are led to study local behavior of nonlinear system near their fixed points, such that one can use them to test trajectories in the Lagrange points of the reduced problem of three bodies.

This is done in linear system, so we can start to study the stability of a fixed point and the geometric properties of trajectories near it.

Hadamard has used the geometric properties to construct the unstable manifold of diffeomorphism of the plane, and we remark that the analytical methods have historical les use, but recently appeared many new methods and algorithms that have been advanced to compute unstable manifold such as have been used in space mission in order to make them more fuel efficient. Lastly all these motivate the desire to find the existence of solution for non linear system.

1.1 Preliminaries:

Let us first define adynamical system as an ordinary differential equation,

$\frac{dx}{dt} = f\ (x, t)$. Where $x \in R^n$

$f : R^n \times R \rightarrow R^n$ is a vector field.We can define f on subset

$E \subset R^n$. Then the domain of f is called (phase space) of the system. Our focus will be on systems where f does not depend on time (called autonomous system) if f vanishes, $x(t) = x(0)$, it is called (equilibrium points) of the system, and to study these points we can go back to the notion of invariant set which we can define now.

1.2 Important Classes of Invariant Mainfolds:

Since the invariant manifolds are extensions of eigen space, these have similar names with similar meaning. Let us give some technical definition of stable and unstable manifolds,

An invariant set is a subset A ⊂ E of the phase space such that for any

$x \in$ A and t $\in R, \phi_t(x) \in$ A, where $\phi_t(x)$ is the flow .

In particular we will study invariant sets associated with equilibrium points. If we let P be such a point we define the stable and unstable sets w^s and w^u respectively, associated with P, as follow.

Definition (1): The stable eigenspace E^s is the space spanned by the eigenvectors whose corresponding eigenvalues have negative real parts.

Definition (2): The unstable eigenspace E^u is the space spanned by the eigenvectors whose corresponding eigenvalues have positive real parts.

Definition (3): (The stable points). The stable manifold w^s of an equilibrium point is a set of points in phase space with the following two properties:

a) For $x \in w^s$,$\varphi^t(x) \rightarrow p$as t $\rightarrow \infty$

b) w^s is tangent to E^s at P.

Definition (4): (The unstable points). The unstable manifold w^u of an equilibrium point is a set of points in phase space with the following two properties:

a) For $x \in w^u$, $\varphi^t(x) \rightarrow p$ as t $\rightarrow \infty$

b) w^u is tangent to E^u at P.

Definition (5): A set S $\subset R^{n+m}$ is an invariant manifold for the equation

$$\begin{Bmatrix} \dot{x} = \mathrm{Ax} + \mathrm{f}(\mathrm{x},\mathrm{y}) \\ \dot{y} = \mathrm{By} + \mathrm{g}(\mathrm{x},\mathrm{y}) \end{Bmatrix} \quad (1)$$

If for any solution $(\mathrm{x}(\mathrm{t}), \mathrm{y}(\mathrm{t})), (\mathrm{x}(0), \mathrm{y}(0)) \in \mathrm{S}$ .We have that for some $\mathrm{T} > \mathrm{o}$, $(\mathrm{x}(\mathrm{t}), \mathrm{g}(\mathrm{t})) \in$S for all t$\in [\mathrm{o}, \mathrm{T}]$.

Here we restrict our attention to invariant manifold for (1) which are tangent at the origin to the invariant subspacey $=$ o for the corresponding linear problem.

$$\begin{Bmatrix} \dot{x} = \mathrm{A\,x} \\ \dot{y} = \mathrm{B\,y} \end{Bmatrix} \quad (2)$$

Definition (6): The centre eigenspace E^s is the space spanned by the eigenvectors whose corresponding eigenvalues have a real part of zero.

Definition (7): (The centre manifold). The centre manifold of an equilibrium point P is an invariant manifold of the differential equations with the added property that the manifold is tangent to E^s at P.

Definition (8): An invariant manifold S = $\{(\mathrm{x}, h(\mathrm{x})) \| \mathrm{x} \| < \delta\}$ for (1) is a centre manifold if $h(\mathrm{o}) = \mathrm{o}$, $\mathrm{D}h(\mathrm{o}) = 0$.

Theorem (1): In some neighborhood u of the equilibrium point there exist unique centre manifold w^c such that for any$x \in$ u

$\varphi^t(t) \rightarrow \mathrm{W}^c$ as $t \rightarrow \infty$

Theorem (2): There exists a center manifold $h(x)$,$|\mathrm{x}| > \delta$ for (1) of class C^k the flow on this manifold is governed by the n-dimensional system as in $\dot{u} = A\,u + f\,(u, h, (u))$ (3)

1.3 Some applications:

Now we turn to compute the stable and unstable sets in case of linear system which is realized by

$$\dot{x} = Ax \quad (*)$$

A is a matrix n $\times$ n we can start to solve by using

$\exp(\mathrm{A}) = \sum_{\mathrm{i}=0}^{\infty} \frac{1}{i!} A^i$,and the matrixnorm by

$\|\mathrm{A}\| = sup_{x=|x|=1}|A\,x|$and it holds inequality$max_{j,k}[\mathrm{A_{j,k}}] \leq \|\mathrm{A}\| \leq \mathrm{n}\,max_{j,k}[\mathrm{A_{\,j,k}}]$ so that the formal sum of exponential converges in the matrix norm if every entry converges, so $\|A^i\| \leq \|A\|^i$

Proof: $\|A^i\| = sup_{x=|x|=1}|A^i x| = sup_{x=|x|=1}|A^{i-1}\ (Ax)| \leq$

$sup_{x=|x|=1}|Ax| sup_{x=|x|=1}|Ax| \leq \|A\| \|A^{i-1}\|$and the conclusion follows by induction, thus we have convergence of the matrix exponential from the convergence of $\sum_{i=0}^{\infty} \frac{1}{i!} \|A\|^i$then the direct differentiation shows that, we can see

$x(\mathrm{t}) = \exp(tA)x(\mathrm{o}) = \sum_{\mathrm{i}=0}^{\infty}\frac{\mathrm{t}^{i}\mathrm{A}^{i}}{i\,!}x(\mathrm{o})$is a solution of $(*)$with initial condition$x\,(o)$.Now we notice that the origin is a fixed point of any such system, and these leads to study its stable and unstable sets which turn out to beline or subspace of R^n. If we restrict to the case $= 2$.

Furthermore we assume the eigenvalues of A have non zero real part.

Remark: Any linear system that satisfies the condition $x(\mathrm{o})$is called (hyperbolic) and the origin is ahyperbolic fixed Point. So we want to prove that the invariant sets are generalized eigenspace of matrix A and the following holds:

$\mathrm{u}^{-1}A^{j}\mathrm{u} = (\mathrm{u}^{-1}\,\mathrm{Au})^{\mathrm{j}} \Longrightarrow \mathrm{u}^{-1}\exp\,(A)\mathrm{u} = \exp(\mathrm{u}^{-1}\,\mathrm{Au})$.

In system $(*)$we perform a change of coordinate so A is an Jordan form is done in C^2 rather than R^2 , so the solution are constrained to vectors with zero imaginary part, let A has two distinct eigen values it has Jordan from

$$A = \begin{pmatrix} \propto_1 & 0 \\ 0 & \propto_2 \end{pmatrix} \Rightarrow \exp(A) = \begin{pmatrix} e^{\propto_1}\, 0 & \\ 0 & e^{\propto_2} \end{pmatrix}$$

So in new coordinates

$x\,(0) = x_{0,1}u_1 + x_{0,2}u_2 \Longrightarrow x\,(t) = e^{t\propto_1}x_{0,1}u_1 + e^{t\propto_2}x_{0,2}u_2$ (4)

Where$u_1, u_2$are eigenvectors of $\propto_1$ and$\propto_2$, assume that $\propto_1 = \propto_2{}^*$ and $u_2 = u_1{}^*$ since $\mathrm{x}(0)^* = x\,(0)$ for real initial condition we get

$x*_{0,2} = x_{0,1}$, the equation (4) becomes$x\,(t) = 2\,R_e\,(e^{t\propto_1}x_{0.1}u_1)$.

In both cases and to check that

$\left.\begin{matrix} W^s = E^s \\ W^u = E^u \end{matrix}\right\}$ E^s and E^u are direct sum of generalized eigenspaces of eigenvales with negative respectively positive real part.

On other case when there existsonly one eigenvalue, we have

$A = \begin{pmatrix} \propto & 1 \\ 0 & \propto \end{pmatrix}$ and then $\exp(tA)\begin{pmatrix} t\propto & te^{t\propto} \\ 0 & e^{t\propto} \end{pmatrix}$

Proof: $A^j = \begin{pmatrix} \propto^j\; j\propto^{j-1} & \\ 0 & \propto^j \end{pmatrix}$, it holds for $j = 1$

$$A^{j+1} = \begin{pmatrix} \propto^j\; j\propto^{j-1} & \\ 0 & \propto^j \end{pmatrix}\begin{pmatrix} \propto & 1 \\ 0 & \propto \end{pmatrix} = \begin{pmatrix} \propto^{j+1}\;(j+1)\propto^j & \\ 0 & \propto^{j+1} \end{pmatrix}$$

Then we get $x(0) = x_{0,1}u_1 + x_{0,2}u_2$ which implies that

$\mathrm{x}(\mathrm{t}) = (\mathrm{e}^{\mathrm{t}\propto}x_{0.1} + \mathrm{t}\,\mathrm{e}^{\propto}\mathrm{x}_{0.2})u + e^{t\propto}x_{0.2}u_2$, since $\propto$ is real, select u_1 and u_2 to be real, the solution is valid for real initial conditions.

Clearly: $W^s = R^n$ if $\propto < o$, $W^u = R^n$ if $\propto > o$ are some in first case its true.

Theorem (3): In the system given by the equation $(*)$.The eigen values of A have non zero real parts.

$W^s = E^s$

$W^u = E^u$

In particular $R^n = W^s \oplus W^u$

1.4 Non linear system:

After we know about stable and unstable manifold, let us apply in non linear system. First we start by the following theorem

1.4.1 Stable and unstable manifold:

Suppose the origin is a fixed point of $\dot{x} = (x)$ .E^s and E^u subspaces of linearization $\dot{x} = Ax$, where $A = Df\,(x)$ is the Jacobian of origin

If $|f(x) - A(x)| = o(|x^2|)$ then there exist a local stable and unstable manifolds W^s_{loc} (o), W^u_{loc} (o) which have same dimension asE^s and E^s are tangent to them at o such that for $x \in$ u, $x \neq o$ for some neighborhood$u$ of O

$W^s_{loc}(o) = \{x = \phi_t(x) \to o \ \ as \ \ t \to \infty$

$W^u_{loc}(o) = \{x = \phi_t(x) \to o \ \ as \ \ t \to \infty$

This manifold can be extend and called global manifolds by letting

$W^s(o) = \{x: \exists\, t \in R, \ s.t \ \ \phi_t(x) \in W^s_{loc}(o)\}$

$W^u(o) = \{x: \exists\, t \in R, \ s.t \ \ \phi_t(x) \in W^u_{loc}(o)\}$

Example:

$\dot{x} = -x + y^2, \dot{y} = y - x^2$

The origin is a fixed point of this system and the linearization is given by $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ and we have

$|f(x) - Ax| = \left|\begin{matrix} y^2 \\ -x^2 \end{matrix}\right| = |x^2|$we can apply the theorem first, notice that

$E^u = \text{span}\begin{pmatrix}0\\1\end{pmatrix}$

$E^u = \text{span}\begin{pmatrix}1\\0\end{pmatrix}$

To get approximation of solution for manifolds first we look at unstable manifold its tangent to $x = o$,let $x = p(y)$ we get

$$-p(y) + y^2 = \dot{x} = \acute{p}(y)\dot{y} = \acute{p}(y)\left(y - (p(y))^2\right) \qquad (5)$$

to complete the solution, using a power series, Since we need p(y) to be tangent to $x = o$ so the first two terms vanish in

$p(y) = a_2 y^2 + a_3 y^3 + \cdots$

Plugging this back in to (5) we can match coefficient and get aseries for p(y)in this problem only, we can use athird order approximate solution is $W^u_{loc}(o) = \left\{x: x = \begin{pmatrix}\frac{1}{3}y^2\\ y\end{pmatrix}\right\}$ we can use the same method to stable manifold also gives $W^s_{loc}(o) = \left\{x: x = \begin{pmatrix}x\\ \frac{1}{3}x^2\end{pmatrix}\right\}$

This example is illustrated how difficulties can arise in computing the manifolds of two dimensional systems.

1.5 The Jacobian eigenvalues:

The Jacobian is play important role in variant manifold as in vicinity of equilibrium point x^* .

Now let us to show these work, assume x^* is point of equuilibrium of autonomous of (ODE) $\dot{x} = f(x)$ (6)

Provided that none of the eigenvalues are zero, the (DE)can be approximated by $\dot{\delta x} = J^*\delta x$ (7)

Where $\delta x = x - x^*$ and J^* is Jacobian evaluated at x^*, J^* is a constant so (7) is a linear (ODE) and it's solution is given $\delta x(t) = \sum c_i e_i e^{\lambda_i t}$ where$e_i \equiv$ right eigen vectors, and λ_i is corresponding eigen values of $J^*$and c_i are some coefficiend select to satisfy the initial conditions.

1.6 The flow dynamical equilibrium point:

Let x_1 is arbitrary point in the phase space of equation (6) and let x_2 denote to second point close to x_1 and δx be a vector connecting $x_1 \neq x_2$, so $\delta x = x_2 - x_1$ and the evolution ofδx is

$\dot{\delta x} = \dot{x}_2 - \dot{x}_1 = f(x_2) - f(x_1) = f(x_1 + \delta x) - f(x)$.

Taylor can extend$f(x_1 + \delta x)$, about $\delta x = o$:

$f(x_1 + \delta x) = f(x_1) + J_1\,\delta x + \cdots$Here$J_1$ is Jacobian evaluated at x_1. The lowest order the difference vector δx obeys the differential equation $\dot{\delta x} \approx J_1 \delta x$.

1.7 Applications of invariant manifold:

The equations of Lindemann mechanism are

$$\left.\begin{array}{l}\dot{a} \;=\; -\ a^2 + \propto a\ b \\ \dot{b} \;=\; a - \propto ab - b\end{array}\right\} \Longrightarrow F\,(a,b) = (f_1(a,b), f_1(a,b))$$

$a' = f_1(a,b),\ b' = f_1\ (a,b)$. And the Jacobian at the equilibrium point (0, 0) was $J^* = \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}$

The eigenvalues $\lambda_0 = 0, \lambda_1 = -1$ this is a system of centre manifold.

First we determine the eigenvectors of J^* which satisfy

$J^* e_i = \lambda_i e_i$, *or* $(\lambda_i I - J^*)e_i = 0$ *For* t_o, *we get* $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}e_{01}\\ e_{02}\end{bmatrix} = 0$

That $C_{02} = o,\ e_o = (0,1)$ similary for λ_1

$\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}e_{i1}\\ e_{i2}\end{bmatrix} = 0$ which implies that e_1 = (0, 1) and the two eigenvectors in this case happen to be the two coordinate axes. In sure not generally true, the case of the eigenvectors are orthogonal to each other.

Theorem (4): (i) Suppose that the zero solution of (3) is astable, then if $[x(t), y(t)]$ is a solution of (1) with $(x\ (o), y\ (o))$sufficiently small there is a solution $u(t)$ of (3) such that for $t \geq 0$,

$$x\ (t) = u(t) +\ \varphi_1\ (t)$$

$$y(t) = h\big(u(t)\big) +\ \varphi_2\ (t)$$

With $\varphi_1(t)$ and $\varphi_2(t) \to o$ as $\to \infty$

(ii) Suppose the zero solution of(3) is instable, then zero solution of (1) is unsalable.

Theorem (5): *If* $\phi(o) = 0, D\phi(o) = 0$ *and* $N(\phi)(x) = o(|x|^2), x \to 0$

For some $q > 1$ *then* $|h(x) - \phi(x)| = 0\ (|x|^2), x \to 0$

Example: Consider the second equation

$\ddot{w} + \dot{w} + w^3 = 0$ (8)

Where $w \in R$, setting $x = w + \dot{w}, = \dot{w}$. We may restate (3) as the first order system

$$\left.\begin{array}{l}\dot{x} = (x-y)^3 \\ \dot{y} = -y - (x - y^3)\end{array}\right\} \qquad (9)$$

This in the canonical from (3), the corresponding eigenvalues being 0 and -1, so by theorem (2) there is a centre manifold $y = h(x), |x| < \delta$. To approximate h we set

$N(\phi)(x) = -\dot{\phi}(x)(x - \phi(x))^3 + \phi(x) + (x - \phi(x))^3$ (10)

For $(\phi)(x) = -x^3$, we have $N(\phi)(x) = 0(|x|^5)$. There is theorem (5)

$h(x) = -x^3 + 0(|x|^5)$, and by theorem (4) the equation which determine asymptotic behavior of small solution of (9) is

$\dot{u} = -\big(u - h(u)\big)^3 = -(u + u^3 + 0(|x|^5))^3$

$= -u^3 - 3u^5 + 0(|x|^7)$. (11)

Since the zero solution of this equation (11) is asymptotically stable, by the theorem (2) the zero solution of (9) is also asymptotic stable . By deeper study of the asymptotic behavior for (11) so theorem (2) may be used once again to show that a given solution of (9) either tends to zero exponentially fast with time or has the form

$$x(t) = \pm\, u(t)\,, \qquad y(t) = \pm u^3(t) (12)$$

in which

$u(t) = \frac{1}{\sqrt{2}} t^{-½} - \frac{3}{4\sqrt{2}} t^{-1½}\, logt + ct^{-1½} + 0(t^{-1½})$ (13)

For some constant c.

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